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On the quantum theory of the damped harmonic oscillator

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Abstract. The longstanding problem of the physical interpretation of the one-dimensional quantum damped oscillator is analysed here from the viewpoint of group theory. It is shown how to transform the system into a quantum oscillator with variable frequency. However, the main features of the final system are rather those of a simple stationary oscillator with 'renormalised' constant frequency. The quantum mechanical system cannot be at all interpreted as a 'dissipative' quantum system as has been claimed by some authors.

1. Introduction

Consider the classical motion of the one-dimensional damped harmonic oscillator, described by the equation

$$m\,\mathrm{d}^2x/\mathrm{d}t^2+b\,\mathrm{d}x/\mathrm{d}t+m\omega_0^2x=0. \tag{1}$$

As is well known, one can derive (1) from the so-called Bateman Lagrangian (Dekker 1981) which has the following form:

$$\mathscr{L}=\exp(bt/m)[\tfrac{1}{2}m(\mathrm{d}x/\mathrm{d}t)^2-\tfrac{1}{2}m\omega_0^2x^2]=(\tfrac{1}{2}m\omega_0^2)\,\mathrm{e}^{\gamma_0\tau}(\dot{x}^2-x^2). \tag{2}$$

Here, $\dot{x}=\mathrm{d}x/\mathrm{d}\tau$ where $\gamma_0=b/m\omega_0$ and $\tau=\omega_0t$ are dimensionless. Indeed, one can also construct, by using standard methods, a Hamiltonian which reads

$$H=\mathrm{e}^{-\gamma_0\tau}p^2/2m+\tfrac{1}{2}m\omega_0^2\,\mathrm{e}^{\gamma_0\tau}x^2 \qquad (p=m\omega_o\,\mathrm{e}^{\gamma_0\tau}\dot{x}). \tag{3}$$

It can easily be seen that (3) is not conserved (i.e. $\mathrm{d}H/\mathrm{d}t\neq 0$). Proceeding in an 'orthodox' manner, one can try to go into quantum mechanics by replacing $p\to\mathrm{i}\hbar\,\partial/\partial x$ and then solving the time-dependent Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\exp(-\gamma_0\omega_0t)\frac{\partial^2}{\partial x^2}+\tfrac{1}{2}m\omega_0^2\exp(\gamma_0\omega_0t)x^2\right)\psi(x,t)=\mathrm{i}h\frac{\partial}{\partial t}\psi(x,t). \tag{4}$$

This is the Caldirola–Kanai equation (Dekker 1981). Although this mathematical problem seems to be easy to solve, the main question concerning the physical interpretation of the system from the quantum mechanical viewpoint remains somewhat obscure. Does (4) represent a quantum system describing *dissipation*, as (3) usually represents a classical mechanical system with *friction*? Is the phenomenological description of damping valid at the quantum level? These and other related questions have been a matter of debate for more than forty years. They have been answered in the affirmative (Caldirola 1941, 1983, Dodonov and Manko 1978, 1979) and also in the negative (Senitzky 1960, Greenberger 1979). There also exists an alternative interpretation of

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the Caldirola–Kanai equation (4) as describing a quantum system with exponentially varying mass (Colegrave and Abdalla 1981, Leach 1983), as well as a fairly large amount of literature considering several other aspects of this Hamiltonian in the quantum domain. For a review see Dekker (1981).

In this paper, we shall try to clarify the issue, but hope not to add more confusion to that already existing. In order to do so, we start from a point of view which has been overlooked by other authors, namely, group theory. In a preceding paper (Cerveró and Villarroel 1984) we have analysed the groups leaving the equation of motion (1) unchanged and the action (2) invariant, the latter being a subgroup of the former. This large group of symmetry has several interesting properties: it allows for a time transformation (still a canonical transformation) which leads us to contemplate the Lagrangian (2) as that of a harmonic oscillator with *variable frequency* and with a very specific (and unique) time dependence for the *frequency*. The theory of the harmonic oscillator with variable frequency has been considered by a large number of authors as far as the invariance group and constants of motion are concerned (Prince and Eliezer 1980, Leach 1980, Takayama 1982, Colegrave and Abdalla 1983). Also coherent states have been constructed for this system very recently (Hartley and Ray 1982). We shall use many of these results and we shall extend and generalise some of them in order to apply them to our specific problem. The conclusion will be, roughly speaking, that due to the large group of symmetry, group theory forces us to a unique consistent physical interpretation of the Caldirola–Kanai equation. This interpretation is not consistent either with dissipation (loss energy states) or with variable mass. It must only be interpreted as the *only variable frequency oscillator with minimum uncertainty*. This statement will be clarified at the end of the paper.

The plan of this paper is as follows. In § 2 we shall revise our previous results in group theory of the damped harmonic oscillator. Section 3 is devoted to a similar group theoretical analysis of the harmonic oscillator with variable frequency. In § 4 we show briefly the standard quantisation of the Caldirola–Kanai Hamiltonian and we shall recover the well known coordinate-dependent wavefunctions. In § 5 we revise the quantum theory of the harmonic oscillator with variable frequency and special attention is paid to the coherent state representation for this case. The transformation from damping to variable frequency is carried out in § 6 which also contains the physical interpretation, the main features of our quantum mechanical system and the conclusions. We close with an appendix dealing with the contact symmetry group of the variable frequency classical oscillator: a novel result that we have found as a byproduct of our group theoretical considerations.

2. Group theory of the damped harmonic oscillator

This section summarises the results obtained by us in a preceding paper (Cerveró and Villarroel 1984). The symmetry group of (1) is SL(3, R) and consists of the following eight infinitesimal generators:

$$G_1=\frac{2}{\gamma}[\sinh(\gamma\tau)\frac{\partial}{\partial\tau}+\left(\frac{\gamma}{2}\cosh(\gamma\tau)-\frac{\gamma_0}{2}\sinh(\gamma\tau)\right)x\frac{\partial}{\partial x}\Big],$$

$$G_2=\frac{2i}{\gamma}\left[\cosh(\gamma\tau)\frac{\partial}{\partial\tau}+\left(\frac{\gamma}{2}\sinh(\gamma\tau)-\frac{\gamma_0}{2}\cosh(\gamma\tau)\right)x\frac{\partial}{\partial x}\right],$$

$$G_3 = (2i/\gamma)[\exp(-\gamma_0\tau/2)\cosh(\tfrac{1}{2}\gamma\tau)\partial/\partial x],$$
$$G_4 = (2/\gamma)[\exp(-\gamma_0\tau/2)\sinh(\tfrac{1}{2}\gamma\tau)\partial/\partial x], \quad (5)$$
$$G_5 = (2i/\gamma)(\partial/\partial\tau - \tfrac{1}{2}\gamma_0 x\,\partial/\partial x), \qquad G_6 = x\,\partial/\partial x,$$
$$G_7 = -i\,e^{\gamma_0\tau/2}\left\{x\sinh\left(\frac{\gamma\tau}{2}\right)\frac{\partial}{\partial\tau} + \left[\frac{\gamma}{2}\cosh\left(\frac{\gamma\tau}{2}\right) - \frac{\gamma_0}{2}\sinh\left(\frac{\gamma\tau}{2}\right)\right]x^2\frac{\partial}{\partial x}\right\},$$
$$G_8 = e^{\gamma_0\tau/2}\left\{x\cosh\left(\frac{\gamma\tau}{2}\right)\frac{\partial}{\partial\tau} + \left[\frac{\gamma}{2}\sinh\left(\frac{\gamma\tau}{2}\right) - \frac{\gamma_0}{2}\cosh\left(\frac{\gamma\tau}{2}\right)\right]x^2\frac{\partial}{\partial x}\right\},$$

where $\gamma = (\gamma_0^2 - 4)^{1/2}$. Only $\{G_1, G_2, G_3, G_4$ and $G_5\}$ are symmetry generators of the action (2). Through Noether's theorem we obtain five invariants, associated to these generators:

$$I_1 = e^{\gamma_0\tau}\{(\dot{x}^2 + x^2)\sinh(\gamma\tau) - x\dot{x}[\gamma\cosh(\gamma\tau) - \gamma_0\sinh(\gamma\tau)] + \tfrac{1}{2}x^2[\gamma^2\sinh(\gamma\tau) - \gamma\gamma_0\cosh(\gamma\tau)]\},$$
$$I_2 = e^{\gamma_0\tau}\{(\dot{x}^2 + x^2)\cosh(\gamma\tau) - x\dot{x}[\gamma\sinh(\gamma\tau) - \gamma_0\cosh(\gamma\tau)] + \tfrac{1}{2}x^2[\gamma^2\cosh(\gamma\tau) - \gamma\gamma_0\sinh(\gamma\tau)]\},$$
$$\left.\begin{aligned} I_3 &= e^{\lambda_2\tau}(\dot{x} + \lambda_1 x) \\ I_4 &= e^{\lambda_1\tau}(\dot{x} + \lambda_2 x)\end{aligned}\right\} \quad \lambda_1 = \frac{\gamma_0 - \gamma}{2}, \qquad \lambda_2 = \frac{\gamma_0 + \gamma}{2}, \quad (6)$$
$$I_5 = e^{\gamma_0\tau}(\dot{x}^2 + x^2 + \gamma_0 x\dot{x}).$$

We also obtain the following relations among them:

$$I_1 = \tfrac{1}{2}(I_3^2 - I_4^2), \qquad I_2 = \tfrac{1}{2}(I_3^2 + {}_4^2), \qquad I_5 = I_3 I_4. \quad (7)$$

Since the action is invariant under G_5, we easily see that this generator is a non-conventional Hamiltonian (not a Legendre transform but certainly conserved) as opposed to the conventional one (Legendre transformed but not conserved). Let us consider G_5 as a vector field associated to the system (2). Thus, we have

$$d\tau = -\frac{2}{\gamma_0}\frac{dx}{x} = -\frac{2}{\gamma_0}\frac{d\dot{x}}{\dot{x}} = \frac{2}{\gamma_0}\frac{dp}{p}$$

where we have used $p = m\omega_0\exp(\gamma_0\tau)\dot{x}$. The first integrals of the above differential system are

$$Q = \exp(\gamma_0\tau/2)x, \qquad P = \exp(-\gamma_0\tau/2)p, \quad (8)$$

and $\{Q, P\} = 1$. The resulting Hamiltonian after performing this canonical transformation is

$$H^* = P^2/2m + \tfrac{1}{2}m\omega_0^2 Q^2 + \tfrac{1}{2}\omega_0\gamma_0 QP. \quad (9)$$

It is trivial to see that (9) is the invariant I_5 in $(\tfrac{1}{2}m\omega_0^2)$-units:

$$H^* = (\tfrac{1}{2}m\omega_0^2)[e^{\gamma_0\tau}(\dot{x}^2 + x^2 + \gamma_0 x\dot{x})]. \quad (10)$$

3. Group theory of the harmonic oscillator with variable frequency

The equations of motion are

$$d^2x/dt^2 + \Omega^2(t)x = 0 \tag{11}$$

and the Lagrangian per unit mass, has the form

$$\tilde{\mathscr{L}} = \tfrac{1}{2}(dx/dt)^2 - \tfrac{1}{2}\Omega^2(t)x^2. \tag{12}$$

The group leaving (11) invariant was recently found (Leach 1980, Prince and Eliezer 1980) and the infinitesimal generators have the form

$$\begin{aligned}
&F_1 = \rho^2 \sin 2\theta\, \partial/\partial t + x(\rho\dot{\rho} \sin 2\theta + \cos 2\theta)\, \partial/\partial x,\\
&F_2 = \rho^2 \cos 2\theta\, \partial/\partial t + x(\rho\dot{\rho} \cos 2\theta - \sin 2\theta)\, \partial/\partial x,\\
&F_3 = \rho \cos \theta\, \partial/\partial x, \qquad F_4 = \rho \sin \theta\, \partial/\partial x,\\
&F_5 = \rho^2\, \partial/\partial t + \rho\dot{\rho} x\, \partial/\partial x, \qquad F_6 = x\, \partial/\partial x,\\
&F_7 = x\rho \sin \theta\, \partial/\partial t + x^2(\dot{\rho} \sin \theta + \rho^{-1} \cos \theta)\, \partial/\partial x,\\
&F_8 = x\rho \cos \theta\, \partial/\partial t + x^2(\dot{\rho} \cos \theta - \rho^{-1} \sin \theta)\, \partial/\partial x,
\end{aligned} \tag{13}$$

and they close the SL(3, R) Lie algebra if the auxiliary variable $\rho(t)$ is the solution of the Pinney equation (Pinney 1950)

$$\ddot{\rho} + \Omega^2(t)\rho = \rho^{-3} \tag{14}$$

and $\theta(t)$ is defined as

$$\theta(t) = \int_0^t ds\, \rho^{-2}(s). \tag{15}$$

It is interesting to notice that the Lie algebra closed by the F's in (13) is SL(3, R) independently of whether we can find a solution of (14) or not. Only the differential equation (14) has to be used in order to recover the rules for this Lie algebra.

The group leaving the action invariant is formed by the subgroup $\{F_1, F_2, F_3, F_4$ and $F_5\}$ of (13). These generators yield the following Noëther invariants:

$$\begin{aligned}
&J_1 = \tfrac{1}{2}[(x\dot{\rho} - \dot{x}\rho)^2 \sin 2\theta - x^2\rho^{-2} \sin 2\theta + 2x\rho^{-1}(x\dot{\rho} - \dot{x}\rho) \cos 2\theta],\\
&J_2 = \tfrac{1}{2}[(x\dot{\rho} - \dot{x}\rho)^2 \cos 2\theta - x^2\rho^{-2} \cos 2\theta - 2x\rho^{-1}(x\dot{\rho} - \dot{x}\rho) \sin 2\theta],\\
&J_3 = [(x\dot{\rho} - \dot{x}\rho) \cos \theta - x\rho^{-1} \sin \theta],\\
&J_4 = [x\rho^{-1} \cos \theta + (x\dot{\rho} - \dot{x}\rho) \sin \theta],\\
&J_5 = \tfrac{1}{2}[(x\dot{\rho} - \dot{x}\rho)^2 + x^2\rho^{-2}],
\end{aligned} \tag{16}$$

and they also verify

$$J_1 = J_3J_4, \qquad J_2 = \tfrac{1}{2}(J_3^2 - J_4^2), \qquad J_5 = \tfrac{1}{2}(J_3^2 + J_4^2). \tag{17}$$

4. Quantum mechanics of the damped harmonic oscillator

The conventional procedure for quantising the classical system (2) is, as we have already said, to solve the time-dependent Schrödinger equation (the Caldirola–Kanai equation) arising from the choice of (3) as the conventional Hamiltonian operator. This method leads to well known wavefunctions (Dodonov and Manko 1978, 1979), which have several interesting features commonly interpreted as describing 'loss energy' states by the authors that support the idea that (4) represents a true physical quantum dissipative system. The difficulties arising from this interpretation have been extensively discussed. A masterly account has been given by Greenberger (1979). The main drawback concerns the uncertainty principle which has the form

$$(\Delta x)(\Delta p) \geqslant \tfrac{1}{2}\hbar \exp(-\gamma_0 \tau) \tag{18}$$

and turns out to be vanishingly small for large times. We think that this difficulty is only the tip of the iceberg. All problems really come from the ambiguity related to a judicious choice of the Hamiltonian. The Hamiltonian (3) is not conserved since it is not included in the set of invariants (6). A conserved 'Hamiltonian' exists as a constant of motion but it turns out to be not a Legendre transform from Lagrangian (2) and hence it is not related to a symplectic form of the mechanical system described by this Lagrangian. This non-conventional 'Hamiltonian' is obviously H^* given by (9)–(10) and related to the I_5 invariant. However, if we select it as a true Hamiltonian, we can find the solution of the *stationary* problem

$$H^*|\psi_n\rangle = \varepsilon_n|\psi_n\rangle \tag{19}$$

which has also been solved (Dodonov and Manko 1978) with the following curious result:

$$\begin{aligned} \varepsilon_n &= \tfrac{1}{2}\hbar\omega_0\gamma(n+\tfrac{1}{2}), \\ \psi_n(Q) &= \frac{1}{(2^n n!)^{1/2}}\left(\frac{\alpha}{\pi}\right)^{1/4} \exp\left\{\left[\left(\frac{\gamma_0 - \mathrm{i}\gamma}{4}\right)\omega_0 - \frac{\mathrm{i}n\omega_0\gamma_0}{2}\right]t - \frac{\mathrm{i}m\omega_0\gamma_0}{4\hbar}Q^2\right\} \\ &\quad \times \exp(-\alpha^2Q^2/2)H_n(\alpha Q), \end{aligned} \tag{20}$$

where $\alpha = (m\omega_0\gamma/2\hbar)^{1/2}$, $\gamma = (4-\gamma_0^2)^{1/2}$ and H_n are the Hermite polynomials. Notice that substituting in (20) $Q = x\exp(\frac{1}{2}\gamma_0\tau)$ given by (8) we obtain time-dependent wavefunctions which are exactly the solutions of (4). Therefore, the stationary solutions for H^* are also the time-dependent ones for H, using the canonical transformation (8). Indeed, we know from standard quantum mechanics that given an invariant of the quantum system (in this case H^*) we have

$$\mathrm{d}H^*/\mathrm{d}t = \partial H/\partial t + (1/\mathrm{i}\hbar)[H, H^*] = 0. \tag{21}$$

If $|\psi\rangle$ is a vector state of H

$$\mathrm{i}\hbar\,(\partial/\partial t)|\psi\rangle = H|\psi\rangle \tag{22}$$

we obtain, using (21) (Lewis and Riesenfeld 1969) the following identity:

$$\mathrm{i}\hbar(\partial/\partial t)(H^*|\psi\rangle) = H(H^*|\psi\rangle). \tag{23}$$

The eigenvectors of H are also a subset of solutions of the stationary problem for H^*. In our case, even the relative phase between vector states of both problems is a trivial constant and we recover the above result.

All these coincident features are a clear signal that we are dealing with a simple harmonic oscillator *without* dissipation. This view is supported by the fact that the symmetry group for the damped harmonic oscillator and for the harmonic oscillator, even with variable frequency, are the same. Roughly speaking, the problem has too much symmetry and this large symmetry allows us to see the system from different points of view and, apparently, with different physical interpretations. It is this last point in which mistakes might arise. The physical interpretation is unique and we shall see how this subtle point can be uncovered with the help of the symmetry group itself!. In § 5 we revise the coherent state representation for the variable frequency oscillator in the general case in order to compare with the specific case in which we are interested, later on.

5. The coherent state representation for the time-dependent harmonic oscillator

As has been recently shown (Hartley and Ray 1982), we can construct a coherent set of states for the quantum system described by the Hailtonian

$$\tilde{H} = \tfrac{1}{2}p^2 + \tfrac{1}{2}\Omega^2(t)x^2 \tag{24}$$

arising from (12). One can define $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ operators as

$$\hat{a}(t) = (2\hbar)^{-1/2}[x\rho^{-1} + \mathrm{i}(p\rho - x\dot{\rho})], \tag{25}$$

$$\hat{a}^\dagger(t) = (2\hbar)^{-1/2}[x\rho^{-1} - \mathrm{i}(p\rho - x\dot{\rho})]. \tag{26}$$

Of course, $[\hat{a}(t), \hat{a}^\dagger(t)] = 1$. The operators (25)–(26) are connected to the J_3 and J_4 invariants in (16). The J_5 invariant can be cast in the form

$$J_5 = \hbar[\hat{a}^\dagger(t)\hat{a}(t) + \tfrac{1}{2}] \tag{27}$$

Using (27), the eigenvalue problem for J_5 can be exactly solved just as for the time-independent case. Thus, we have

$$J_5|n, t\rangle = \hbar(n + \tfrac{1}{2})|n, t\rangle, \tag{28a}$$

$$\hat{a}(t)|n, t\rangle = \sqrt{n}|n-1, t\rangle, \qquad \hat{a}^\dagger(t)|n, t\rangle = \sqrt{n+1}|n+1, t\rangle. \tag{28b, c}$$

Defining the phase functions as

$$\alpha_n(t) = -(n + \tfrac{1}{2})\int_0^t \mathrm{d}s\,\rho^{-2}(s), \tag{29}$$

the general solution of the time-dependent Schrödinger equation for H in (24) is (Lewis and Riesenfeld 1969)

$$|\psi, t\rangle_s = \sum_n c_n \exp[\mathrm{i}\alpha_n(t)]|n, t\rangle \tag{30}$$

and the coherent states are given by (Hartley and Ray 1982)

$$|\alpha, t\rangle_s = \exp(-|\alpha|^2/2)\sum_n \frac{\alpha^n}{(n!)^{1/2}}\exp[\mathrm{i}\alpha_n(t)]|n, t\rangle \tag{31}$$

where we have used the obvious notation

$$\hat{a}(0)|\alpha, 0\rangle_s = \alpha|\alpha, 0\rangle_s, \qquad |\alpha, 0\rangle_s = \exp(-|\alpha|^2/2)\sum_n \frac{\alpha^n}{(n!)^{1/2}}|n, 0\rangle.$$

Finally, Hartley and Ray also gave the expression for the uncertainty in the coherent state representation for this case of variable frequency:

$$(\Delta x)_\alpha(\Delta p)_\alpha = \tfrac{1}{2}\hbar(\rho^2\dot{\rho}^2+1)^{1/2} \tag{32}$$

which shows that these coherent states are not of minimum uncertainty and they spread in time. We shall find an important exception: exactly our case.

6. The transformation from damping to variable frequency

Let us write (3) in the form

$$H(t)=p^2/2m(t)+\tfrac{1}{2}\omega_0^2 m(t)x^2, \qquad m(t)=m\exp(\gamma_0\omega_0 t). \tag{33}$$

Applying a time transformation of the form (Leach 1983)

$$\bar{t}=\int_0^t m^{-1}(t)=\frac{-1}{m\omega_0\gamma_0}\exp(-\gamma_0\omega_0 t) \tag{34}$$

the new Hamiltonian is ($p=\bar{p}$, $x=\bar{x}$)

$$H(\bar{t})=\tfrac{1}{2}p^2+\tfrac{1}{2}(1/\gamma_0^2\bar{t}^2)x^2. \tag{35}$$

Comparing with (24) we can easily identify

$$\Omega(\bar{t})=(\gamma_0\bar{t})^{-1}. \tag{36}$$

The Pinney equation (14) for this functional form of the frequency is

$$\ddot{\rho}+(1/\gamma_0^2\bar{t}^2)\rho=\rho^{-3} \tag{37}$$

whose general solution takes the form

$$\rho(\bar{t})=[\alpha\bar{t}^{\,\gamma_0+\mathrm{i}\gamma/2\gamma_0}+\beta\bar{t}^{\,\gamma_0-\mathrm{i}\gamma/2\gamma_0}+2(\alpha\beta+\gamma_0^2/\gamma^2)^{1/2}\bar{t}]^{1/2}. \tag{38}$$

One must take $\alpha=\beta=0$ for consistency. Then $\rho(\bar{t})=(2\gamma_0/\gamma)^{1/2}\bar{t}^{1/2}$. The invariant J_5 can easily be found with this form of $\rho(\bar{t})$ and we indeed obtain our previous H^* in the old coordinates. Now, let us apply the results of the quantum harmonic oscillator to this particular time dependence. We easily obtain from (29)

$$\alpha_n(t)=\tfrac{1}{2}\omega_0\gamma(n+\tfrac{1}{2})t \tag{39}$$

which corresponds to a stationary harmonic oscillator (i.e. the simple time-independent oscillator with the frequency 'renormalised' to $\omega_0\gamma/2$). More important is the form of the uncertainty principle (32). We trivially obtain

$$(\Delta x)_\alpha(\Delta p)_\alpha=\hbar/\gamma. \tag{40}$$

Indeed, the harmonic oscillator limit is $\gamma=2$. Therefore, we have for our original 'damped' system: no non-trivial quantum mechanical time dependence (39), no spreading (40) of the coherent states, and of course, no *loss energy* and no *dissipation*. The quantum mechanical system is just a harmonic oscillator with constant frequency equal to $\omega_0\gamma/2$. The spectrum is given by $\varepsilon_n=\tfrac{1}{2}\hbar\omega_0\gamma(n+\tfrac{1}{2})$.

In the past, a large amount of physical information was in need of a rigorous and satisfactory mathematical framework. In this modest example we have encountered an opposite situation: a detailed mathematical analysis forces the physical system to have just one unambiguous physical interpretation.

Appendix. Contact symmetry group for the harmonic oscillator with variable frequency

Contact symmetries seem to be of importance for the total integrability of a classical system. This group of symmetries was recently found for the simple harmonic oscillator (Schwarz 1983) and also for the damped harmonic oscillator (Cerveró and Villarroel 1984). Here, we present a general result concerning such a contact group for the variable frequency harmonic oscillator. The differential equation is

$$\mathrm{d}^2x/\mathrm{d}t^2+\Omega^2(t)x=0 \tag{A1}$$

and we have to solve the contact invariance condition:

$$S_c[q+\Omega^2(t)x]=0 \tag{A2}$$

where

$$\begin{aligned}S_c \equiv \left(\frac{\partial W}{\partial p}\right)\frac{\partial}{\partial t}+\left(p\frac{\partial W}{\partial p}-W\right)\frac{\partial}{\partial x}-\left(\frac{\partial W}{\partial t}+p\frac{\partial W}{\partial x}\right)\frac{\partial}{\partial p}\\ -\left(\frac{\partial^2 W}{\partial t^2}+2p\frac{\partial^2 W}{\partial t\,\partial x}+p^2\frac{\partial^2 W}{\partial x^2}+2q\frac{\partial^2 W}{\partial t\,\partial p}\right.\\ \left.+2qp\frac{\partial^2 W}{\partial x\,\partial p}+q^2\frac{\partial^2 W}{\partial p^2}+q\frac{\partial W}{\partial x}\right)\frac{\partial}{\partial q}\end{aligned} \tag{A3}$$

and $p=\mathrm{d}x/\mathrm{d}t$, $q=\mathrm{d}^2x/\mathrm{d}t^2$. Here, $W\equiv W(x,p,t)$ is the so-called generating function of the contact transformation. The equation (A3) gives rise to a partial differential equation for W:

$$\begin{aligned}\frac{\partial^2 W}{\partial t^2}+2p\frac{\partial^2 W}{\partial t\,\partial x}+p^2\frac{\partial^2 W}{\partial x^2}-2\Omega^2(t)xp\frac{\partial^2 W}{\partial x\,\partial p}\\ +\Omega^4(t)x^2\frac{\partial^2 W}{\partial p^2}-\Omega^2(t)x\frac{\partial W}{\partial x}-2\Omega(t)\dot{\Omega}(t)x\frac{\partial W}{\partial p}\\ +\left(W-p\frac{\partial W}{\partial p}\right)\Omega^2(t)-2\Omega^2(t)x\frac{\partial^2 W}{\partial t\,\partial p}=0.\end{aligned} \tag{A4}$$

Choosing the following set of characteristic curves:

$$u=(x\dot{\rho}-\rho p)\sin\theta+(x/\rho)\cos\theta, \qquad v=(x\dot{\rho}-\rho p)\cos\theta-(x/\rho)\sin\theta, \tag{A5}$$

where ρ and θ verify (14) and (15), one can always change the (x,p) variables to the (u,v) variables. Then we find that (A4) reduces to

$$\partial^2 W/\partial t^2+\Omega^2(t)W=0 \tag{A6}$$

and now, since $W\equiv W(u,v,t)$, the general solution of (A6) can be trivially written down in the form

$$W(u,v,t)=\Delta_1(u,v)\rho\cos\theta+\Delta_2(u,v)\rho\sin\theta. \tag{A7}$$

Therefore, the contact invariance group of (A1) is also (as in all the previous studied cases) an infinite-parameter Lie group.

For specific choices of $\Delta_1(u,v)$ and $\Delta_2(u,v)$ we shall recover the point group already found (Prince and Eliezer 1980, Leach 1980) since the point group is a finite-parameter subgroup of the contact invariance group. These choices are displayed in the table below.

Δ_1	$-u$	$-v$	-1	0	$-v$	$-u$	$-u^2$	$-uv$
Δ_2	$-v$	u	0	-1	$-u$	v	uv	v^2
	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8

In the preceding derivation of the generating function we have made use of the following facts which greatly simplify the calculations. Let us write u and v in the form

$$u = \nu_1 p - \dot{\nu}_1 x, \qquad v = \nu_2 p - \dot{\nu}_2 x. \tag{A8}$$

Then the functions ν_1 and ν_2 verify equation (A1) in the form

$$\ddot{\nu}_1 + \Omega^2(t)\nu_1 = \ddot{\nu}_2 + \Omega^2(t)\nu_2 = 0. \tag{A9}$$

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